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Wehrl's entropy and a measure of intermode correlations in phase space

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Abstract

The properties of the *entangled pure states* in phase space are analysed using the classical entropy introduced by Wehrl. The general entropy inequalities for *non-entangled pure states* are derived, which are violated by any *entangled state*. To measure the strength of *intermode correlations* in phase space the parameters related to these inequalities are proposed. As an example we study the correlations between amplitudes and phases for two-mode Fock states. We find that the amplitudes as well as phases of different modes are correlated. It is also shown that the degree of the *intermode correlation* strongly depends on the photon number difference in two-mode Fock states.

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1. Introduction

Entangled states have become a very attractive subject of study as one of the most striking possibilities in quantum mechanics. Since Schrödinger's paper [1] their nonintuitive features have been analysed in many fundamental works (for instance see [2, 3]). Moreover, many authors focused on the potential applications of entanglement such as quantum computation, quantum cryptography and the teleportation effect ([4, 5] and references quoted therein).

A measure of entanglement usually applied for systems in a pure state is the von Neumann entropy (quantum entropy). A positive value of quantum entropy computed for one of the subsystems is the criterion of entanglement (nonseparability). The general criteria of separability also valid for mixed states were recently proposed and discussed in [6].

In this paper our attention is focused on the phase space representation of pure state entanglement. We propose a term *intermode correlations in a phase space* that can be referred to as entanglement in phase space. To investigate such correlations we introduce definitions that involve the Wehrl entropy concept [7]. In contrast to the von Neumann entropy the latter is a classical quantity characterized by classical entropy properties, and its definition is based on the Husimi Q-function [8]. This function is always positive, in contrast to other quasi-probability distributions defined in phase space for an arbitrary quantum state and, hence, the

Q-function can be used as an analogue of a classical probability distribution in the definition of entropy [9].

The Wehrl entropy has been applied in several works [10–14] to analyse quantum features of one-mode field states. In [15] the concept of a classical entropy was used to define a measure of the quantum phase properties of an optical field. The ideas shown there are similar in a sense to that discussed here. However, we are concerned with multi-mode systems defined in the space that can be split into subspaces, in contrast to the single-mode states discussed in [15].

This paper is organized as follows. We briefly recall the Wehrl entropy concept and its important properties in section 2. In section 3 we derive the information entropy inequalities for a two-mode system valid for all non-entangled pure states. Moreover, we define parameters that can be treated as a measure of the intermode correlation. Section 4 is devoted to the discussion of the two-mode states in the Fock basis in the context of the intermode correlation, in which we focus on the amplitude and phase correlations. As an example the analytical results for the intermode correlations for equally weighted two-component states are derived and discussed in section 5. In addition, we find there the boundary values of the intermode correlations in phase space.

2. Wehrl entropy

The Wehrl entropy [7] has been introduced as a classical approximation of the von Neumann quantum entropy

$$S\big[\hat{\rho}\big] = -\,\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) \tag{1}$$

where $\hat{\rho}$ is the density operator for a given quantum state (the Boltzman constant is taken to be $k = 1$). In analogy to the classical entropy in the phase space, the Wehrl entropy can be written as

$$S\left[\alpha_1, \alpha_2\right] = -\int \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, J\, Q\left(\alpha_1, \alpha_2\right) \ln Q\left(\alpha_1, \alpha_2\right) \tag{2}$$

where $Q\left(\alpha_1, \alpha_2\right)$ is the Husimi function defined with the help of a coherent state $|\alpha\rangle$ as

$$Q\left(\alpha_1, \alpha_2\right) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \tag{3}$$

where

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle \tag{4}$$

and $|0\rangle$ is a vacuum state, $\hat{a}^\dagger$ and $\hat{a}$ denote the creation and the annihilation operators respectively, α is a complex number $\alpha = \alpha_x + \mathrm{i}\alpha_y$ or $\alpha = \alpha_r \mathrm{e}^{\mathrm{i}\alpha_\varphi}$ and $J = 1$ if $(\alpha_1, \alpha_2) = (\alpha_x, \alpha_y)$, or $J = \alpha_r$ if $(\alpha_1, \alpha_2) = (\alpha_r, \alpha_\varphi)$. The definition (2) exploits the unique property of the Q function, which is always positive in contrast to other quantum quasi-probabilities. Hence, Wehrl's entropy is treated as a measure of the lack of information about the system, i.e. as the information entropy of a quantum mechanical state of the system, in analogy to the Shannon formula [7, 9]. From another point of view it can be interpreted as a particular case of sampling entropy [14] related to the operational phase-space measurement proposed by Wódkiewicz [16] (the so-called propensity becomes the Q-function if the filter is taken to be a coherent state in operational phase-space measurement).

The Wehrl entropy, due to its characteristic properties, appears to be very attractive for the study of quantum systems. One of the most important properties is a possibility to distinguish between different pure states, in contrast to the von Neumann entropy, which gives zero for all of them [12]. Additionally, it was proven [7] that

$$S\left[\hat{\rho}\right] \leqslant S\left[\alpha_1, \alpha_2\right] \tag{5}$$

for any state of the system, and the Wehrl entropy reaches its minimum value if the system is in a pure coherent state [7, 17]. The explicit relation between von Neumann and Wehrl entropies was given by Peřinova *et al* [18] (the detailed analysis of the Wehrl entropy can be also found in [7, 19]).

In this paper one of the most important properties of entropy, i.e. its subadditivity, is extensively exploited. Generally, the entropy subadditivity property is expressed by the relation

$$S\left[\sum_i U_i\right] \leqslant \sum_i S\left[U_i\right] \qquad \text{for} \qquad U = \sum_i U_i \tag{6}$$

where U denotes a set of variables describing the system. The equality holds only for independent subsets of variables. In the information theory this formula means that information received from the subsystems cannot be greater than that received from the whole system. Considering the Wehrl entropy of a one-mode state, i.e. $\hat{\rho} = |A\rangle\langle A|$, one can rewrite equation (6) as follows:

$$S\left[\alpha_1, \alpha_2\right] \leqslant S\left[\alpha_1\right] + S\left[\alpha_2\right] \tag{7}$$

where $S\left[\alpha_i\right]$ are the marginal entropies

$$S\left[\alpha_i\right] = -\int \mathrm{d}\alpha_i \; J \; Q\left(\alpha_i\right) \ln Q\left(\alpha_i\right) \tag{8}$$

related to the marginal $Q\left(\alpha_i\right)$ functions

$$Q\left(\alpha_i\right) = \int \mathrm{d}\alpha_j \; J \; Q\left(\alpha_1, \alpha_2\right) \tag{9}$$

with $i, j = 1, 2, i \neq j$, and $J = \alpha_r$ if $\mathrm{d}\alpha_{i,j} = \mathrm{d}\alpha_r$, or $J = 1$ in other cases. The inequality (7) can be interpreted as an entropic uncertainty relation for the variables α_1 and α_2. In general, we receive less information from the marginal $Q(\alpha_i)$ functions than from the joint quasipropability $Q(\alpha_1, \alpha_2)$. The quantum state $|A\rangle$ is called an *intelligent state* if the left and right-hand sides of equation (7) become equal. If, in addition, both sides reach their minimum values, we call the state the *minimum uncertainty state*. The relation (7) was studied extensively for many quantum states [10–14].

To investigate the entropic uncertainty relation for a given quantum state it is convenient to exploit the concept of mutual information $I\left[u, v\right]$ [9] between two variables u and v given by

$$I\left[u, v\right] = S\left[u\right] + S\left[v\right] - S\left[u, v\right]. \tag{10}$$

This quantity measures the information contained in the variable u about the variable v, and vice versa. According to the subadditivity condition (6), the mutual information $I\left[u, v\right]$ takes only non-negative values

$$I\left[u, v\right] \geqslant 0 \tag{11}$$

and it is equal to zero if the variables are independent (there is no information in u about v). Using the joint quasi-probability $Q\left(\alpha_1, \alpha_2\right)$ and the marginal distributions $Q(\alpha_i)$ one can define $I\left[\alpha_1, \alpha_2\right]$ as

$$I\left[\alpha_1, \alpha_2\right] = S\left[\alpha_1\right] + S\left[\alpha_2\right] - S\left[\alpha_1, \alpha_2\right] \tag{12}$$

or in the integral form

$$I\,[\alpha_1, \alpha_2] = \int \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, J\, Q\,(\alpha_1, \alpha_2) \ln \frac{Q\,(\alpha_1, \alpha_2)}{Q\,(\alpha_1)\, Q\,(\alpha_2)}. \tag{13}$$

If $I\,[\alpha_1, \alpha_2] = 0$, the state is called an *intelligent state*. In the other case, $I\,[\alpha_1, \alpha_2] > 0$, there is some information contained in one phase-space coordinate about the second one.

In the next section the subadditivity formula (6) is used as the basis to analyse the *intermode correlations* of states in a multi-mode system.

3. Inter-mode correlation in the Wehrl entropy approach

Let us consider a two-mode state of the field (system) described by the density operator $\hat{\rho}_{AB}$ in the Hilbert space $H = H_A \otimes H_B$. The corresponding quasi-probability Q is determined as

$$Q\,(\alpha_1, \alpha_2; \beta_1, \beta_2) = \frac{1}{\pi^2} \langle \alpha, \beta | \hat{\rho}_{AB} | \alpha, \beta \rangle \tag{14}$$

$|\alpha, \beta\rangle = |\alpha\rangle \otimes |\beta\rangle$. On the basis of this function, the Wehrl entropy of the total system can be written in the form

$$S\,[\alpha_1, \alpha_2; \beta_1, \beta_2] = -\int \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\beta_1\, \mathrm{d}\beta_2\, J\, Q\,(\alpha_1, \alpha_2; \beta_1, \beta_2) \ln Q\,(\alpha_1, \alpha_2; \beta_1, \beta_2). \tag{15}$$

The $Q\,(\alpha_1, \alpha_2; \beta_1, \beta_2)$ function depends on four variables of the phase space. Hence, it allows the six two-parameter quasi-probabilities to be defined as

$$\begin{aligned} Q\,(\alpha_1, \alpha_2) &= \int \mathrm{d}\beta_1\, \mathrm{d}\beta_2\, J\, Q\,(\alpha_1, \alpha_2; \beta_1, \beta_2) \\ Q\,(\beta_1, \beta_2) &= \int \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, J\, Q\,(\alpha_1, \alpha_2; \beta_1, \beta_2) \end{aligned} \tag{16}$$

which represent individual modes, i.e. A or B, in their corresponding phase spaces, and

$$Q\left(\alpha_i; \beta_j\right) = \int \mathrm{d}\alpha_p\, \mathrm{d}\beta_q\, J\, Q\,(\alpha_1, \alpha_2; \beta_1, \beta_2) \tag{17}$$

with $i, j, p, q = 1, 2$ and $i \neq p$, $j \neq q$, called the intermode distributions. Obviously, since they are well defined probabilities, all of them satisfy the normalization condition

$$\int_\Omega Q\,(\Omega)\; J\, \mathrm{d}\Omega = 1 \tag{18}$$

where Ω denotes the corresponding set of variables. Instead of the four-parameter Q function (14) describing the system one can try to use the two-parameter Q functions introduced above. Then, the subadditivity relation (6) can be applied to the system in the following manner:

$$\begin{aligned} S\,[\alpha_1, \alpha_2; \beta_1, \beta_2] &\leqslant S\,[\alpha_1, \alpha_2] + S\,[\beta_1, \beta_2] \leqslant S\,[\alpha_1] + S\,[\alpha_2] + S\,[\beta_1] + S\,[\beta_2] \\ S\,[\alpha_1, \alpha_2; \beta_1, \beta_2] &\leqslant S\,[\alpha_1; \beta_1] + S\,[\alpha_2; \beta_2] \leqslant S\,[\alpha_1] + S\,[\alpha_2] + S\,[\beta_1] + S\,[\beta_2] \\ S\,[\alpha_1, \alpha_2; \beta_1, \beta_2] &\leqslant S\,[\alpha_1; \beta_2] + S\,[\alpha_2; \beta_1] \leqslant S\,[\alpha_1] + S\,[\alpha_2] + S\,[\beta_1] + S\,[\beta_2] \end{aligned} \tag{19}$$

where $S\,[\alpha_1, \alpha_2]$ and $S\,[\beta_1, \beta_2]$ are determined in (2); $S\,[\alpha_i]$, $S\,[\beta_i]$ are the entropies based on the marginal distributions (8) and $S\left[\alpha_i; \beta_j\right]$ are defined as

$$S\left[\alpha_i; \beta_j\right] = -\int \mathrm{d}\alpha_i\, \mathrm{d}\beta_j\, J\, Q\left(\alpha_i; \beta_j\right) \ln Q\left(\alpha_i; \beta_j\right). \tag{20}$$

From inequalities (19) it is seen that there are three different possibilities to choose the two-parameter functions to describe the system. Now, the question arises of which choice gives the best approximation to the information about the total system contained in the joint quasi-probability $Q(\alpha_1, \alpha_2; \beta_1, \beta_2)$, in other words, when the loss of information is minimized.

Let us assume that the two-mode pure state is not entangled. Thus, it can be written in the product form

$$|A, B\rangle = |A\rangle \otimes |B\rangle \tag{21}$$

where $|A\rangle \in H_A$ and $|B\rangle \in H_\mathrm{B}$. The quasi-probability of the entire system

$$Q(\alpha_1, \alpha_2; \beta_1, \beta_2) = \frac{1}{\pi^2}|\langle\alpha, \beta|A, B\rangle|^2 \tag{22}$$

is the product of the single-mode Q functions (16), i.e.

$$\begin{aligned} Q(\alpha_1, \alpha_2; \beta_1, \beta_2) &= \frac{1}{\pi^2}|\langle\alpha|A\rangle|^2|\langle\beta|B\rangle|^2 \\ Q(\alpha_1, \alpha_2; \beta_1, \beta_2) &= Q(\alpha_1, \alpha_2)\, Q(\beta_1, \beta_2)\,. \end{aligned} \tag{23}$$

So, the following entropy relation is obtained:

$$S[\alpha_1, \alpha_2; \beta_1, \beta_2] = S[\alpha_1, \alpha_2] + S[\beta_1, \beta_2]\,. \tag{24}$$

This means that the information on the total system can be extracted without any losses from the Q functions describing individual modes. Hence, these modes are independent systems in the information sense. There are no correlations between variables describing different modes in the phase space. Moreover, inserting equation (23) into formulae (17) and using equation (9), we obtain the following relations:

$$\begin{aligned} Q(\alpha_i; \beta_j) &= Q(\alpha_i)Q(\beta_j) \\ S[\alpha_i; \beta_j] &= S[\alpha_i] + S[\beta_j]. \end{aligned} \tag{25}$$

Hence, the following equalities are satisfied by a state without any correlation:

$$\begin{aligned} S[\alpha_1; \beta_1] + S[\alpha_2; \beta_2] &= S[\alpha_1] + S[\beta_1] + S[\alpha_2] + S[\beta_2] \\ S[\alpha_1; \beta_2] + S[\alpha_2; \beta_1] &= S[\alpha_1] + S[\beta_1] + S[\alpha_2] + S[\beta_2]\,. \end{aligned} \tag{26}$$

In the context of the subadditivity formula (6) it is obvious that the information on the total system represented by the function $Q(\alpha_1, \alpha_2; \beta_1, \beta_2)$ is lost whenever only the $Q(\alpha_i; \beta_j)$ are known. The analysis concluded in equation (24) and (26) provides the entropy inequalities, and the non-entanglement inequalities in phase space that are satisfied for any non-entangled pure state

$$\begin{aligned} S[\alpha_1, \alpha_2] + S[\beta_1, \beta_2] &\leqslant S[\alpha_1; \beta_1] + S[\alpha_2; \beta_2] \\ S[\alpha_1, \alpha_2] + S[\beta_1, \beta_2] &\leqslant S[\alpha_1; \beta_2] + S[\alpha_2; \beta_1]\,. \end{aligned} \tag{27}$$

In general, these inequalities tell us that for all two-mode non-entangled states (21) the information gained from the single-mode Q quasi-probabilities (16) is no less than the information obtained from the intermode distributions (17). Moreover, the equality sign holds if, and only if, the modes are in the intelligent states. If a state violates one of the inequalities (27), or both of them, it can be called a two-mode entangled pure state in the phase space. Here, we have taken into consideration only the two-mode case, but the results can be extended to the multi-mode fields.

To examine the *intermode correlations* (entanglement in the phase space) it is useful to introduce some new parameters. By application of the mutual entropy concept in the form of equation (10), we introduce the following quantities:

$$I[\alpha_i; \beta_j] = S[\alpha_i] + S[\beta_j] - S[\alpha_i; \beta_j] \tag{28}$$

which can be written in the equivalent form

$$I[\alpha_i;\beta_j] = \int \mathrm{d}\alpha_i\,\mathrm{d}\beta_j\, J\, Q(\alpha_i;\beta_j) \ln \frac{Q(\alpha_i;\beta_j)}{Q(\alpha_i)Q(\beta_j)}. \tag{29}$$

They represent a measure of the mutual information between two coordinates associated with the different modes. Taking into account equations (25), the non-correlated variables imply $I[\alpha_i;\beta_j] = 0$, whereas $I[\alpha_i;\beta_j] > 0$ is obtained for any correlation case. We define, in a different way, more general parameters that measure the violation of the non-entanglement inequalities (27)

$$\begin{aligned} L' &= S[\alpha_1,\alpha_2] + S[\beta_1,\beta_2] - (S[\alpha_1;\beta_1] + S[\alpha_2;\beta_2]) \\ L'' &= S[\alpha_1,\alpha_2] + S[\beta_1,\beta_2] - (S[\alpha_1;\beta_2] + S[\alpha_2;\beta_1]) \end{aligned} \tag{30}$$

or

$$\begin{aligned} L' &= \int \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\, J\, Q(\alpha_1,\alpha_2;\beta_1,\beta_2) \ln \frac{Q(\alpha_1;\beta_1)\,Q(\alpha_2;\beta_2)}{Q(\alpha_1,\alpha_2)\,Q(\beta_1,\beta_2)} \\ L'' &= \int \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2\, J\, Q(\alpha_1,\alpha_2;\beta_1,\beta_2) \ln \frac{Q(\alpha_1;\beta_2)\,Q(\alpha_2;\beta_1)}{Q(\alpha_1,\alpha_2)\,Q(\beta_1,\beta_2)}. \end{aligned} \tag{31}$$

For any non-entangled state in phase space the parameters L', L'' are non-positive; i.e., the modes are independent systems in phase space. Otherwise, if both parameters, or one of them, are positive,

$$L' > 0 \qquad \text{or(and)} \qquad L'' > 0 \tag{32}$$

the correlation occurs. So, the relations (32) can be used as a criterion of the intermode correlation in phase space. Applying definitions (12) and (28) we derive the following relationships between L and I parameters:

$$\begin{aligned} L' &= I[\alpha_1;\beta_1] + I[\alpha_2;\beta_2] - (I[\alpha_1,\alpha_2] + I[\beta_1,\beta_2]) \\ L'' &= I[\alpha_1;\beta_2] + I[\alpha_2;\beta_1] - (I[\alpha_1,\alpha_2] + I[\beta_1,\beta_2]). \end{aligned} \tag{33}$$

The L criterion of correlation implies that, for an entangled pure state, the sum of the mutual information shared by the coordinates from different phase spaces (the intermode mutual information) exceeds the sum of the mutual information shared by the variables from single-mode phase spaces (the single-mode mutual information). Additionally, for any non-entangled pure state the parameters L are sums, with negative signs, of the mutual information shared by the single-mode variables, i.e. $L = -I[\alpha_1,\alpha_2] - I[\beta_1,\beta_2]$, and they approach zero if the individual modes are in the intelligent states.

The L parameters, in the forms (30)–(33), allow us to conclude that for any entangled pure state the information carried by the single-mode Q functions (16) is smaller than the information which resides in the intermode quasi-probabilities (17). In other words, if the $Q(\alpha_1,\alpha_2)$ and $Q(\beta_1,\beta_2)$ are measured for a two-mode entangled state, the loss of information about the total system is greater than in determining the intermode distributions $Q(\alpha_i;\beta_j)$. So, the relationship between the phase-space coordinates of different modes is stronger than that between the coordinates describing single modes. The parameters presented here can be used to investigate the intermode correlations of two-mode quantum systems in phase space. We shall pay special attention to the two-mode states in the Fock space.

4. Two-mode states in the Fock basis

It is often convenient to represent quantum states in the Fock number space. We analyse in detail the intermode correlation for the two-mode states in the Fock basis, applying the method proposed in previous sections. It is important to note that the Fock number states exhibit highly non-classical features and are associated with corpuscular properties of the electromagnetic field. To exemplify the correlation description in phase space two types of state are taken into account: (i) a product state

$$|i\rangle = |k_i\rangle_A |l_i\rangle_B \tag{34}$$

and (ii) a state in the Schmidt decomposition

$$|f\rangle = \sum_{i=1}^{N} c_i |k_i\rangle_A |l_i\rangle_B \tag{35}$$

where k_i and l_i denote numbers of photons (particles), $\langle k_i|k_j\rangle = \langle l_i|l_j\rangle = \delta_{ij}$ and $c_i = |c_i|\mathrm{e}^{\mathrm{i}\varphi_i}$, $\sum_{i=1}^{N} |c_i|^2 = 1$. It was proved by Gisin [20] that any Schmidt decomposition state, such as the state in the form (35), violates Bell's inequalities, showing its non-local character. The density operator of this state $\hat{\rho}_f$ can be written as a sum of two factors,

$$\hat{\rho}_f = \hat{\rho}_{\text{mix}} + \hat{\rho}_{\text{int}} \tag{36}$$

where

$$\hat{\rho}_{\text{mix}} = \sum_{i=1}^{N} p_i \; |k_i\rangle_A |l_i\rangle_{BB}\langle l_i|_A\langle k_i| \tag{37}$$

with $p_i = |c_i|^2$, and

$$\hat{\rho}_{\text{int}} = \sum_{i=1;\, j\neq i}^{N} c_i c_j^* \; |k_i\rangle_A |l_i\rangle_{BB}\langle l_j|_A\langle k_j|. \tag{38}$$

The von Neumann measure of entanglement (1) is always greater than zero (the entanglement exists) for such states and is reduced to classical Shannon entropy,

$$S[\hat{\rho}_f] = -\sum_{i=1}^{N} p_i \; \ln p_i. \tag{39}$$

In our approach we focus on the relationships between the amplitudes and the phases, so the coordinates in phase space are specified as $\alpha_1 = \alpha_r$, $\alpha_2 = \alpha_\varphi$, $\beta_1 = \beta_r$ and $\beta_2 = \beta_\varphi$. The joint Q_i quasi-probability of the total system in the product state $|i\rangle$ takes the following form:

$$Q_i(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) = \frac{1}{\pi^2} \mathrm{e}^{-(\alpha_r^2+\beta_r^2)} \; \frac{\alpha_r^{2k_i} \beta_r^{2l_i}}{k_i! l_i!}. \tag{40}$$

The joint distribution for the Schmidt decomposition

$$Q_f(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) = Q_{\text{mix}}(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) + Q_{\text{int}}(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) \tag{41}$$

includes the interference part

$$\begin{aligned} Q_{\text{int}}(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) = \frac{2}{\pi^2} \,\mathrm{e}^{-(\alpha_r^2+\beta_r^2)} \sum_{i=1;\, j>i}^{N} |c_i||c_j| \, \frac{\alpha_r^{k_i+k_j} \beta_r^{l_i+l_j}}{\sqrt{k_i! k_j! l_i! i_j!}} \\ \times \cos\left[\varphi_{ji} + (k_i - k_j)\alpha_\varphi + (l_i - l_j)\beta_\varphi\right] \end{aligned} \tag{42}$$

and the mixed part

$$Q_{\text{mix}}(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) = \frac{1}{\pi^2} \mathrm{e}^{-(\alpha_r^2+\beta_r^2)} \sum_{i=1}^{N} p_i \, \frac{\alpha_r^{2k_i} \beta_r^{2l_i}}{k_i! l_i!} \tag{43}$$

that can be expressed in terms of the quasi-probabilities of the product states

$$Q_{\text{mix}}(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) = \sum_{i=1}^{N} p_i \, Q_i(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi) \tag{44}$$

following the formula (40). This form of the Q_f function suggests that $|f\rangle$ can be treated as a two-mode superposition in phase space. To investigate the correlation between the modes we calculate the two-parameter quasi-probabilities according to equations (16) and (17). The product state quasi-probabilities take the following forms:

$$\begin{aligned} Q_i(\alpha_r, \alpha_\varphi) &= Q_i(\alpha_r; \beta_\varphi) = \frac{1}{\pi} \mathrm{e}^{-\alpha_r^2} \frac{\alpha_r^{2k_i}}{k_i!} \\ Q_i(\alpha_r; \beta_r) &= 4\, \mathrm{e}^{-(\alpha_r^2+\beta_r^2)} \frac{\alpha_r^{2k_i} \beta_r^{2l_i}}{k_i! l_i!} \\ Q_i(\alpha_\varphi; \beta_\varphi) &= \frac{1}{4\pi^2}. \end{aligned} \tag{45}$$

On replacing $\alpha \leftrightarrow \beta$ and the numbers of photons $k_i \leftrightarrow l_i$, we obtain the $Q_i(\beta_r, \beta_\varphi)$, $Q_i(\beta_r; \alpha_\varphi)$ distributions. For the Schmidt decomposition state the phase joint function only, among all of the two-parameter quasi-distributions, includes the interference part, i.e.

$$Q_f(\alpha_\varphi; \beta_\varphi) = Q_{\text{mix}}(\alpha_\varphi; \beta_\varphi) + Q_{\text{int}}(\alpha_\varphi; \beta_\varphi) \tag{46}$$

where

$$\begin{aligned} Q_{\text{mix}}(\alpha_\varphi; \beta_\varphi) &= Q_i(\alpha_\varphi; \beta_\varphi) \\ Q_{\text{int}}(\alpha_\varphi; \beta_\varphi) &= \frac{1}{2\pi^2} \sum_{i=1; j>i}^{N} C_{ij} \, |c_i||c_j| \cos\left[\varphi_{ji} + (k_i - k_j)\alpha_\varphi + (l_i - l_j)\beta_\varphi\right] \end{aligned} \tag{47}$$

with $\varphi_{ji} = \varphi_j - \varphi_i$ and

$$C_{ij} = G(k_i, k_j)\, G(l_i, l_j) \tag{48}$$

where

$$G(m, n) = \frac{\Gamma\left((m+n)/2 + 1\right)}{\sqrt{m!n!}} \tag{49}$$

is the coefficient introduced in [21, 22]. Other two-parameter distributions take the forms reduced to the mixed parts only:

$$Q_f\,(u, v) = Q_{\text{mix}}\,(u, v) \tag{50}$$

that are weighted sums of the product state distributions

$$Q_{\text{mix}}\,(u, v) = \sum_{i=1}^{N} p_i \, Q_i\,(u, v) \tag{51}$$

(except for the case of the phase joint function). Hence, the differences between the Schmidt decomposition state and the corresponding mixed state (37) manifest themselves in the form of its phase joint function only. The similarities originate from the disappearance of the Q_{int} term when the integration over the phase is performed, $\int_0^{2\pi} \mathrm{d}\alpha_\varphi \, Q_{\text{int}} = 0$, for example. To study the Schmidt decomposition state in more detail we determine the marginal distributions. After the integration over the phase the same results for this state and the corresponding mixed state are obtained:

$$Q_f\,(\alpha_r) = Q_{\text{mix}}\,(\alpha_r) \tag{52}$$

where

$$\begin{aligned} Q_{\mathrm{mix}}\,(\alpha_r) &= \sum_{i=1}^{N} p_i\; Q_i\,(\alpha_r) \\ Q_i\,(\alpha_r) &= 2\,\mathrm{e}^{-\alpha_r^2}\,\frac{\alpha_r^{2k_i}}{k_i\,!} \end{aligned} \tag{53}$$

and

$$Q_i(\alpha_\varphi) = Q_{\mathrm{mix}}(\alpha_\varphi) = Q_f(\alpha_\varphi) = \frac{1}{2\pi} \tag{54}$$

i.e., the phase marginal distributions are flat. To express the marginal Q functions for the B mode, the exchange of $k_i \leftrightarrow l_i$ and $\alpha \leftrightarrow \beta$ is to be performed. It is clearly seen that there are no differences between the Schmidt decomposition and the corresponding mixed states whenever they are represented by their marginal Q quasi-probabilities. In conclusion, the relations between the two-parameter and the marginal distributions can be shown explicitly for the product state

$$Q_i\,(u, v) = Q_i\,(u)\,Q_i\,(v) \tag{55}$$

which is an obvious result in the context of the general non-entangled state analysis of section 2, and for the Schmidt decomposition and the corresponding mixed states

$$\begin{aligned} &Q_{f(\mathrm{mix})}(\alpha_r; \beta_r) \neq Q_{f(\mathrm{mix})}(\alpha_r) Q_{f(\mathrm{mix})}(\beta_r) \\ &Q_f(\alpha_\varphi; \beta_\varphi) \neq Q_f(\alpha_\varphi) Q_f(\beta_\varphi) \\ &Q_{\mathrm{mix}}(\alpha_\varphi; \beta_\varphi) = Q_{\mathrm{mix}}(\alpha_\varphi) Q_{\mathrm{mix}}(\beta_\varphi) \end{aligned} \tag{56}$$

and

$$Q_{f(\mathrm{mix})}\,(u, v) = Q_{f(\mathrm{mix})}\,(u)\,Q_{f(\mathrm{mix})}\,(v) \tag{57}$$

for the remaining cases.

Thus, in figures 1 and 2 we plot $Q_f(\alpha_r; \beta_r)$ for a special case of the state represented by the two-mode Schmidt decomposition, analysed in section 4, i.e. the equally weighted two-component states: $N = 2$, $|c_1| = |c_2| = 1/\sqrt{2}$. Figure 1 shows $Q_f(\alpha_r; \beta_r)$ for various values of the difference of the number of photons $k = k_1 - k_2$, for $l_1 = k_2$, $l_2 = k_1$ and $k_2 = 1$. Two peaks in the α_r, β_r plane, seen in figure 1, become more pronounced with increasing values of $|k|$. For $k = 0$ the picture with a single peak occurs, which is characteristic for the product state, according to $Q_f(\alpha_r; \beta_r) = Q_f(\alpha_r) Q_f(\beta_r)$. In figure 2 the same dependence is visualized for $l_1 = k_1$, $l_2 = k_2$. In the latter case one of the peaks is always smaller than the other one. For the same type of state as in figure 1, in figure 3 the function $Q_f(\alpha_\varphi; \beta_\varphi)$ is plotted for a few values of k. With increasing photon-number difference $|k|$, this function tends to the flat distribution, as for the $k = 0$ case (product state). For the states with $l_1 = k_1$ and $l_2 = k_2$, one can obtain similar pictures that differ only in the angle.

On insertion of expressions (56) and (57) into equations (12) and (28), we find the single-mode mutual information for the radius and phase coordinates

$$I_{f(i)}[\alpha_r, \alpha_\varphi] = I_{f(i)}[\beta_r, \beta_\varphi] = 0 \tag{58}$$

and the intermode mutual information

$$\begin{aligned} &I_f[\alpha_r; \beta_\varphi] = I_f[\alpha_\varphi; \beta_r] = 0 \\ &I_{f(\mathrm{mix})}[\alpha_r; \beta_r] > 0 \\ &I_f[\alpha_\varphi; \beta_\varphi] > 0. \end{aligned} \tag{59}$$

Strictly speaking, the relations (59) prove a correlation between the amplitudes of fields for the Schmidt decomposition state and the corresponding mixed state, whereas the phase

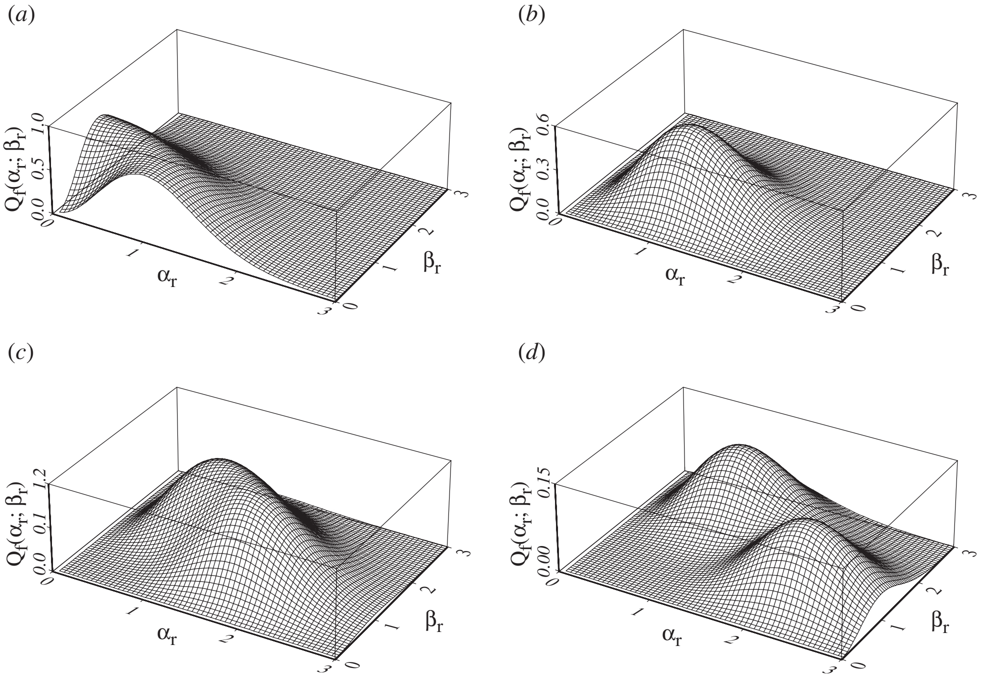

Figure 1. Plot of the amplitude joint quasi-probability $Q_f(\alpha_r; \beta_r)$ for the photon-number Schmidt decomposition state $|f\rangle$ with two equally weighted components, $|c_1| = |c_2| = 1/\sqrt{2}$, and $k_1 = l_2$ and $k_2 = l_1$ for $k_2 = 1$ and (*a*) $k = -1$, (*b*) $k = 0$, (*c*) $k = 2$ and (*d*) $k = 4$, where $k = k_1 - k_2$.

correlation only for the former. However, only the Schmidt decomposition state should be treated as entangled (analysis is concerned with pure states). For more general description the entanglement measure L, given by (30), is suitable. From the relationships (33) the following expressions for the L parameters are obtained:

$$L'_f = I_f[\alpha_r; \beta_r] + I_f[\alpha_\varphi; \beta_\varphi] \tag{60}$$

whereas

$$L''_f = 0. \tag{61}$$

The result (61) means that there are no informational differences, in the Wehrl entropy sense, between the decomposition into the intermode $Q_f(\alpha_r; \beta_\varphi)$ and $Q_f(\alpha_\varphi; \beta_r)$ distributions, and the decomposition into the single-mode Q functions when considering the Schmidt states in the Fock basis (35). Because of positive values of the mutual entropies the parameters L'_f in equation (60) are always positive. So, according to our general consideration in section 2, this fact proves a correlation between the modes connected to the amplitude and the phase correlations. Additionally, taking into account the definition of the parameter L', the equations (60) show explicitly that the information carried simultaneously by the intermode quasi-probabilities $Q_f(\alpha_r; \beta_r)$ and $Q_f(\alpha_\varphi; \beta_\varphi)$ exceeds the information obtained from the single-mode phase space distributions $Q_f(\alpha_r, \alpha_\varphi)$ and $Q_f(\beta_r, \beta_\varphi)$. Hence, the knowledge of the single-mode Q functions implies the loss of information about the total two-mode system, i.e. the Schmidt decomposition state, greater than the knowledge of intermode Q functions.

Moreover, the information about the entire two-mode system described by the $Q_{\text{mix}}(\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi)$ is complete for the corresponding mixed state (37) when the amplitude

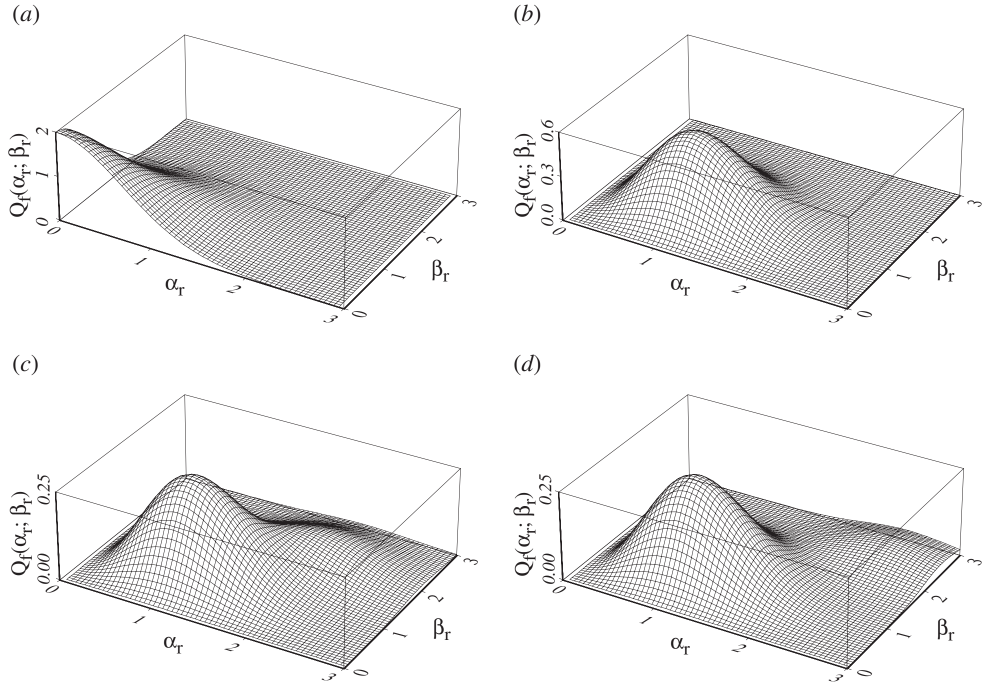

Figure 2. Plot of the amplitude joint quasi-probability $Q_f\,(\alpha_r;\beta_r)$ for the photon-number Schmidt decomposition state $|f\rangle$ with two equally weighted components, $|c_1| = |c_2| = 1/\sqrt{2}$, and $k_1 = l_1$ and $k_2 = l_2$ for $k_2 = 1$ and (*a*) $k = -1$, (*b*) $k = 0$, (*c*) $k = 3$ and (*d*) $k = 5$, where $k = k_1 - k_2$.

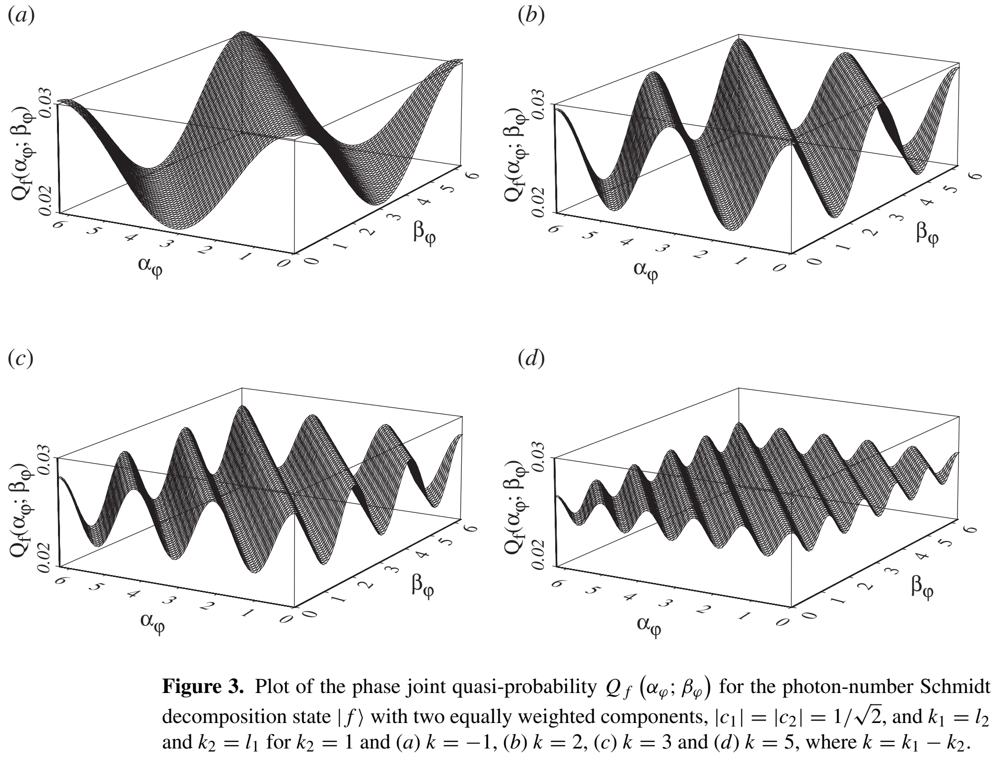

Figure 3. Plot of the phase joint quasi-probability $Q_f\left(\alpha_\varphi;\beta_\varphi\right)$ for the photon-number Schmidt decomposition state $|f\rangle$ with two equally weighted components, $|c_1| = |c_2| = 1/\sqrt{2}$, and $k_1 = l_2$ and $k_2 = l_1$ for $k_2 = 1$ and (*a*) $k = -1$, (*b*) $k = 2$, (*c*) $k = 3$ and (*d*) $k = 5$, where $k = k_1 - k_2$.

joint and the phase joint quasi-probabilities are found,

$$S_{\text{mix}}[\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi] = S_{\text{mix}}[\alpha_r; \beta_r] + S_{\text{mix}}[\alpha_\varphi; \beta_\varphi]. \tag{62}$$

The equalities (60) and (62) immediately imply that

$$S_{\text{mix}}[\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi] > S_f[\alpha_r, \alpha_\varphi; \beta_r, \beta_\varphi] \tag{63}$$

due to the purity of the Schmidt decomposition in the two-mode space.

5. Two-component states

As an example, we shall briefly analyse the correlations in phase space for two-component states ($N = 2$) in the Schmidt decomposition (35). For simplicity equally weighted states are assumed, i.e. $c_i = 2^{-1/2}\,\mathrm{e}^{\mathrm{i}\varphi_i}$. Then, the mutual information (28) between the phases takes the form

$$\begin{aligned} I_f[\alpha_\varphi; \beta_\varphi] &= \frac{1}{4\pi^2}\int_0^{2\pi} \mathrm{d}\alpha_\varphi \int_0^{2\pi} \mathrm{d}\beta_\varphi\, f(\alpha_\varphi; \beta_\varphi) \ln f(\alpha_\varphi; \beta_\varphi) \\ f(\alpha_\varphi; \beta_\varphi) &= 1 + C_{12}\, \cos\left(\varphi_{21} + (k_1 - k_2)\alpha_\varphi + (l_1 - l_2)\beta_\varphi\right) \\ C_{12} &= G(k_1, k_2)\, G(l_1, l_2) \end{aligned} \tag{64}$$

for $k_1 \neq k_2$ and $l_1 \neq l_2$, whereas the mutual information between amplitudes can be written as follows:

$$\begin{aligned} I_f[\alpha_r; \beta_r] &= \ln 2 - \int_0^{\infty} \mathrm{d}\alpha_r \int_0^{\infty} \mathrm{d}\beta_r\, \alpha_r \beta_r\, Q_f(\alpha_r; \beta_r) \ln\left(1 + f(\alpha_r; \beta_r)\right) \\ Q_f(\alpha_r; \beta_r) &= 2\,\mathrm{e}^{-(\alpha_r^2+\beta_r^2)} \left(\frac{\alpha_r^{2k_1}\beta_r^{2l_1}}{k_1!l_1!} + \frac{\alpha_r^{2k_2}\beta_r^{2l_2}}{k_2!l_2!}\right) \\ f(\alpha_r; \beta_r) &= \frac{\alpha_r^{2k_1}\beta_r^{2l_2}k_2!l_1! + \alpha_r^{2k_2}\beta_r^{2l_1}k_1!l_2!}{\alpha_r^{2k_1}\beta_r^{2l_1}k_2!l_2! + \alpha_r^{2k_2}\beta_r^{2l_2}k_1!l_1!}. \end{aligned} \tag{65}$$

The double integral in equation (65) can take only positive values and reaches its lower limit equal to zero for the maximal correlation between amplitudes, and its upper limit equal to $\ln 2$ if $|k_1 - k_2| = 0$ or (and) $|l_1 - l_2| = 0$, implying $|f\rangle \to |i\rangle$. This result is consistent with the general property of mutual information $I[u, v]$ (11). Hence, the mutual information between the amplitudes for the Schmidt decomposition state (65) has its upper bound equal to $\ln 2$. A similar result can be calculated for more than two components in the equally weighted two-mode superposition

$$\max\left(I_f[\alpha_r; \beta_r]\right) = \ln N. \tag{66}$$

It is interesting that the von Neumann entropy criterion for the pure state entanglement (1), i.e. the entropy of the density operator reduced to one of the subsystems, gives the same value,

$$S\left[\hat{\rho}_{A(B)}\right] = \ln N \tag{67}$$

for any equally weighted Schmidt decomposition state with N components, independently of the photon (particle) number in the modes. The same result is also obtained for the photon number index of correlation [23]. In contrast to these quantities the L parameters are sensitive to the number of photons in the entangled modes (figures 4 and 5). We realize that the parameters introduced here do not obey all mathematical criteria required for a measure of entanglement; however, we believe that L parameters could be applied to studies of entanglement similarly to the von Neumann entropy parameters.

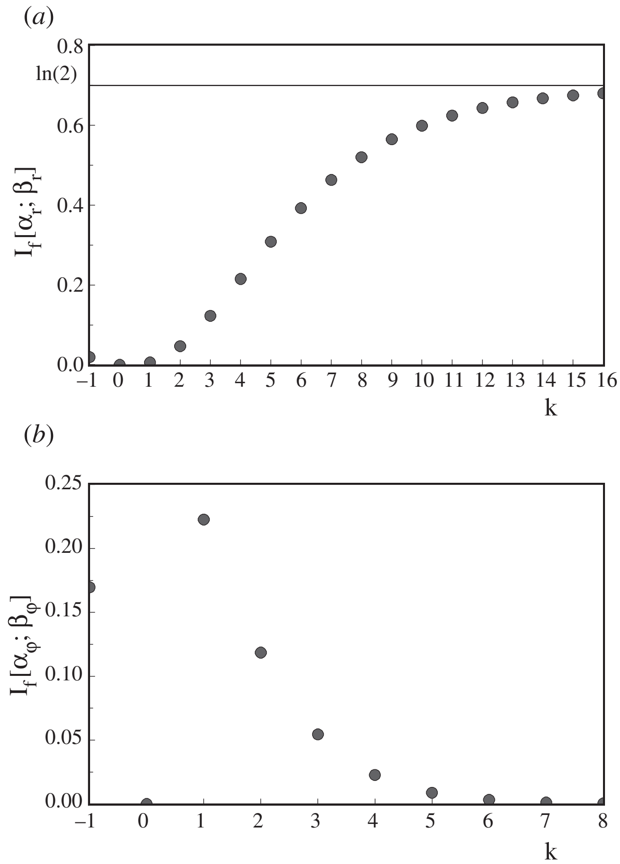

Figure 4. The mutual information of amplitudes $I[\alpha_r; \beta_r]$ (*a*) and the mutual information of phases $I[\alpha_\varphi; \beta_\varphi]$ (*b*) for the state $|f\rangle_2$ as functions of k, where $k = k_1 - k_2$, for $k_2 = 1$.

Here, we restrict our further consideration to a special class of the two-component Schmidt decomposition states, namely

$$|f\rangle_1 = \frac{1}{\sqrt{2}}\left(e^{i\varphi_1}|k_1\rangle_A|k_1\rangle_B + e^{i\varphi_2}|k_2\rangle_A|k_2\rangle_B\right) \tag{68}$$

and

$$|f\rangle_2 = \frac{1}{\sqrt{2}}\left(e^{i\varphi_1}|k_1\rangle_A|k_2\rangle_B + e^{i\varphi_2}|k_2\rangle_A|k_1\rangle_B\right). \tag{69}$$

These states can be treated as a photon number analogue to the states in the EPR effect [2]. Such systems and their non-classical features have been examined theoretically and experimentally and are found in many applications in quantum cryptography, quantum computation or quantum teleportation. For these states the mutual information for the amplitudes can be expressed, with regard to the special form of functions in equation (65), as follows: for $|f\rangle_1$

$$\begin{aligned} Q_f(\alpha_r; \beta_r) &= 2\,e^{-(\alpha_r^2+\beta_r^2)}\left(\frac{(\alpha_r\beta_r)^{2k_1}}{k_1!^2} + \frac{(\alpha_r\beta_r)^{2k_2}}{k_2!^2}\right) \\ f(\alpha_r; \beta_r) &= \frac{\alpha_r^{2k_1}\beta_r^{2k_2} + \alpha_r^{2k_2}\beta_r^{2k_1}}{\alpha_r^{2k_1}\beta_r^{2k_1}d + \alpha_r^{2k_2}\beta_r^{2k_2}d^{-1}} \end{aligned} \tag{70}$$

and for $|f\rangle_2$

$$Q_f(\alpha_r;\beta_r) = 2\,\mathrm{e}^{-(\alpha_r^2+\beta_r^2)}\frac{1}{k_1!k_2!}\left(\alpha_r^{2k_1}\beta_r^{2k_2}+\alpha_r^{2k_2}\beta_r^{2k_1}\right)$$
$$f(\alpha_r;\beta_r) = \left(\frac{\alpha_r^{2k_1}\beta_r^{2k_2}+\alpha_r^{2k_2}\beta_r^{2k_1}}{\alpha_r^{2k_1}\beta_r^{2k_1}d+\alpha_r^{2k_2}\beta_r^{2k_2}d^{-1}}\right)^{-1} \tag{71}$$

where $d = k_2!/k_1!$. The function $f(\alpha_\varphi;\beta_\varphi)$ appearing in formula (64) for the mutual entropy of phases has a very similar form for both states under study

$$f(\alpha_\varphi;\beta_\varphi) = 1 + G^2(k_1,k_2)\,\cos(\varphi_{21} + (k_1-k_2)(\alpha_\varphi \pm \beta_\varphi)) \tag{72}$$

where the sign '+' holds for $|f\rangle_1$ and '$-$' for $|f\rangle_2$ state. The values of $I_f[\alpha_r;\beta_r]$ and $I_f[\alpha_\varphi;\beta_\varphi]$ depend on the particular number of photons and are strongly sensitive to the photon number difference in the component states. We find that the mutual entropy between the amplitudes reaches its maximum value equal to ln 2 in the large-'distance' limit $|k_1-k_2| \gg 1$, that is

$$I_f[\alpha_r;\beta_r] = \ln 2. \tag{73}$$

On the other hand, the mutual entropy between the phases goes to zero in the same limit

$$I_f[\alpha_\varphi;\beta_\varphi] = 0. \tag{74}$$

Moreover, it is completely independent of the choice of the phase shift values φ_{21}. In figure 4(a), we present the mutual entropy between the amplitudes $I_f[\alpha_r;\beta_r]$ for the state $|f\rangle_2$ as a function of the photon number difference $k = k_1 - k_2$. Quite similar behaviour can be obtained for $|f\rangle_1$. In the large-$|k|$ limit these differences disappear and the intermode mutual information of amplitudes reaches ln 2. For the same k-dependence as in figure 4(a), the mutual information between the phases $I_f[\alpha_\varphi;\beta_\varphi]$ takes the same values for both states, as seen in figure 4(b). This function decreases rapidly for large $|k|$. In figure 5 we plot $I_f[\alpha_r;\beta_r]$ and $I_f[\alpha_\varphi;\beta_\varphi]$ for the particular values of k_2 with a fixed 'distance', $k = 1$. With increasing k_2 the mutual information between the amplitudes decreases and reaches zero for large values, but, in contrast, $I_f[\alpha_\varphi;\beta_\varphi]$ increases at the same time.

Following results (70)–(72), we calculate a more general correlation parameter L'_f given by (60). It is clear that the main contribution to its value originates from the amplitude dependences $I_f[\alpha_r;\beta_r]$ when the difference between the number of photons increases. So, in the limit of large photon number difference, L'_f reaches its maximum, which is the upper bound for the correlation,

$$L'_f = \ln 2 \tag{75}$$

for both $|f\rangle_1$ and $|f\rangle_2$ states. In contrast, $I_f[\alpha_\varphi;\beta_\varphi]$ contributes to the entanglement parameter L'_f significantly for a small 'distance' $|k_1-k_2|$ and its contribution increases for large values of k_2, as seen from figure 5(b).

6. Conclusions

We have studied the intermode correlations in phase space (that we refer to as entanglement in phase space) of pure states in composite quantum systems by exploiting their Q-function representations and the Wehrl entropy concept. Applying the subadditivity relation for the entropy, the inequalities, valid for any two-mode non-entangled product state, have been derived. To measure the strength of the violation of the inequalities new parameters (L' and L'') have been introduced, which can be expressed in terms of the mutual information. A positive value of one of them proves a correlation between the subsystems, whereas a negative value or zero means non-correlated states. Moreover, a positive value of the parameter L

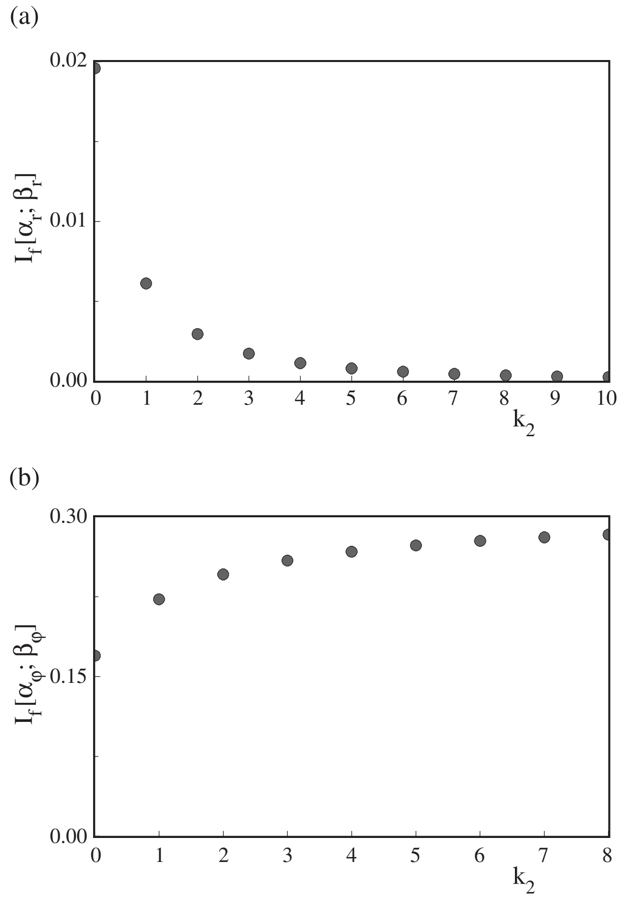

Figure 5. The mutual information of amplitudes $I[\alpha_r; \beta_r]$ (*a*) and the mutual information of phases $I[\alpha_\varphi; \beta_\varphi]$ (*b*) for the state $|f\rangle_2$ as functions of k_2 for $k = 1$, where $k = k_1 - k_2$.

implies that the information contained in the intermode Q-functions exceeds the information obtained from the single-mode quasi-probabilities, showing stronger dependences between the phase-space coordinates of different modes than that for individual modes. Then the single-mode Q-functions appear to be a less informative description of the two-mode system when compared with the intermode distributions. We also want to emphasize that the pure state correlation criteria proposed in this paper are quasiclassical; i.e. they are based on the Wehrl entropy approximating the quantum entropy (1) and obey classical entropy properties.

As an example, we have considered the two-mode states in the Fock basis, focusing our attention on the pure states in the Schmidt decomposition. The amplitude and phase correlations have been investigated with the help of the corresponding Q-functions. We have concluded that the amplitudes of different modes are correlated, in the Wehrl entropy sense, both for the Schmidt decomposition and for the corresponding mixed states, whereas the phase correlation exists only for the former. It has been also shown that there are no relationships between the amplitudes and phases of different modes. In fact, only the coordinates describing the Schmidt decomposition in phase space (the amplitudes and phases) are entangled. A detailed discussion has been given for the two-component equally weighted Schmidt decomposition states. We have derived the upper bound for the mutual entropy (correlation) between the amplitudes, which is equal to ln 2. It is easy to generalize this result for N components in the superposition, which results in the value ln N. Considering a special class of the above-discussed states we have found that the correlation measure (L') strongly depends on the difference in the photon number in component states. With an increase in this variable, the two characteristic peaks that appear in the joint amplitude Q-function become more pronounced, whereas the phase joint quasi-probability tends to a uniform distribution. Accordingly, the mutual information of the amplitudes increases and reaches the value ln 2 in the large-photon-number-difference limit, while the mutual information of the phases decreases and reaches zero in the same limit. For such a case the differences between the Schmidt decomposition state and the corresponding mixed state disappear in the mutual entropy approach but the correlation exists for the former ($L'_f > 0$). Quite the opposite, if the photon number difference is fixed and small, when compared to the photon number in the component states, the mutual information of the amplitudes becomes less important than the phase mutual information in the correlation parameter (so the differences between the Schmidt decomposition state and the corresponding mixed state are significant even in the mutual entropy description).

We have shown that the mutual information between the amplitudes for the equally weighted Schmidt decomposition state gives the same values as the von Neumann entropy criterion for the pure state entanglement (1), and for the photon number index of the correlation [23]. In contrast to these quantities the L parameters are sensitive to the number of photons in the correlated modes. Although the parameters introduced here do not obey all mathematical criteria required for a measure of entanglement, we believe that L parameters could be applied to study entanglement analogously to the von Neumann entropy parameters.

Studying correlations in the Schmidt decomposition states, we have also found that whenever any phase integration is performed the interference term in the Q function disappears. Hence, if the single-mode quasi-probabilities are under consideration the state in the Schmidt decomposition looks like the corresponding mixed state. This also suggests that any effect dependent on the single-mode phase, such as single-mode squeezing, cannot occur in this type of two-mode state.

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